Representations of Products of Groups

$$H := G_1 \times G_2$$

$f_1 \in L^c(G_1)$, $f_2 \in L^c(G_2)$

$\rightsquigarrow$ $f = f_1 \otimes f_2 \in L^c(H)$ (or $f_1 f_2$)

by $f((g_1, g_2)) := f_1(g_1) f_2(g_2)$

Prop: $f_1, f_1' \in L^c(G_1)$, $f_2, f_2' \in L^c(G_2)$

$$\langle f_1 \otimes f_2 , f_1' \otimes f_2' \rangle_H = \langle f_1, f_1' \rangle_{G_1} \cdot \langle f_2, f_2' \rangle_{G_2}$$

$H = G_1 \times G_2$

Theorem: If χ_1 = character of a G_1-rep and χ_2 = character of a G_2-rep.

then $\chi_1 \otimes \chi_2$ = character of a $\underline{H}$-rep

If $\chi_\psi = \chi_{\varphi_1} \otimes \chi_{\varphi_2}$, we call ψ the tensor product representation of φ_1 and φ_2 and write $\psi = \varphi_1 \otimes \varphi_2$

Note: $\dim \psi = (\dim \varphi_1)(\dim \varphi_2)$

Cor: $H = G_1 \times G_2$

$\chi_1, \ldots, \chi_r$ - all irreducible characters of G_1.

$\chi_1', \ldots, \chi_s'$ - all irreducible characters of G_2

Then $\{ \chi_i \otimes \chi_j' \mid 1 \le i \le r,\ 1 \le j \le s \}$

is a complete set of irreducible characters of H.

Proof: Easy to show $\{ \chi_i \otimes \chi_j' \}$ is an orthonormal subset of $L^c(H)$

$\hookrightarrow$ basis

$\dim L^c(H) =$ number of conjugacy classes in H

$= r \cdot s$

Rcm: $H = G \times G \supseteq \Delta = \{ (g,g) \mid g \in G \} \approx G$

so if χ_1, χ_2 - characters on G.

$\Rightarrow \quad \chi := \mathrm{Res}^H_\Delta \, \chi_1 \otimes \chi_2 \in L^c(G)$

have $\quad \chi(g) = \chi_1(g)\chi_2(g)$

$\Rightarrow \quad X(G) \subseteq L^c(G)$

characters

$\rightarrow \quad \underbrace{X(G)}_{\text{"commutative semi-ring"}}$ is closed under

and under

and $0, 1 \in \quad)$

0-rep $\quad$ triv. ρ.

Existence of tensor product representations:

$$G_1 \xrightarrow{\varphi} GL(V) \qquad \overset{??}{\Longrightarrow} \qquad \underset{G_1 \times G_2}{\overset{\parallel}{H}} \xrightarrow{\rho} GL(U)$$

$$G_2 \xrightarrow{\psi} GL(W)$$

Choose bases:

$v_1, \dots, v_m$ of V $\Longrightarrow$ $\varphi_x(v_j) = \sum_{i=1}^{m} \underline{\varphi_{ij}(x)} v_i$.

$w_1, \dots, w_n$ of W $\Longrightarrow$ $\psi_y(w_j) = \sum_{i=1}^{n} \underline{\psi_{ij}(y)} w_i$.

Let U = vector space of $\dim = mn$

with basis $\{ u_{i,j} \mid 1 \le i \le m, \ 1 \le j \le n \}$

Define $\rho : \underset{\substack{\shortparallel \\ G_1 \times G_2}}{H} \longrightarrow GL(U)$

$$\rho_{(x,y)}(u_{j,l}) = \sum_{i=1}^{m} \sum_{k=1}^{n} \underline{\varphi_{ij}(x)}\, \underline{\psi_{kl}(y)}\, u_{i,k}$$